Common Fixed-Point Theorems for Four Mappings in Fuzzy Metric Space Using Rational Inequality

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Abstract

The present paper deals with some common fixed-point results in a fuzzy metric space using rational inequality.

Key words: Co-incidence point, weakly compatible maps, Occasionaly weakly compatible maps, fuzzy metric space.

Introduction

Zadeh [19] introduced the concept of fuzzy sets in 1965. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [12] and George and Veeramani [6] modified the notion of fuzzy metric spaces with the help of continuous t-norm, which shows a new way for further development of analysis in such spaces. Consequently in due course of time some metric fixed points results were generalized to fuzzy metric spaces by various authors. Sessa [16] improved cmmutativity condition in fixed point theorem by introducing the notion of weakly commuting maps in metric space.Vasuki [18] proved fixed point theorems for R-weakly commuting mapping Pant [14,15] introduced the new concept of reciprocally continuous mappings and established some common fixed point theorems. The concept of compatible maps by [3] and weakly compatible maps by [3] in fuzzy metric space is generalized by A.Al Thagafi and Naseer Shahzad [3] by introducing the concept of occasionally weakly compatible mappings. Recent results on fixed point in fuzzy metric space can be viewed in[1,2,].
In this paper we prove some fixed point theorems for four occasionally weakly compatible mappings which improve the result of C.T. Aage[1,2].

Definitions

- **Lemma 2. 1 :** Let $(X,M,*)$ be fuzzy metric space. If there exists $q\in(0,1)$ such that $M(x,y,qt)\geq M(x,y,t)$ for all $x,y\in X$ and $t>0$, then $x=y$.
- **Definition 2. 2 :** Let X be a set, f and g selfmaps of X. A point $x\in X$ is called a coincidence point of f and g iff $fx=gx$. We shall call $w=fx=gx$ a point of coincidence of f and g.
- **Definition 2. 3** [3] **:** A pair of maps S and T is called weakly compatible pair if they commute at coincidence points. The concept of occasionally weakly compatible is introduced by A. Al- Thagafi and Naseer Shahzad [3].It is stated as follows.
- **Definition 2. 4 :** Two self maps f and g of a set X are called occasionally weakly compatible (owc) iff there is a point $x\in X$ which is a coincidence point of f and g at which f and g commute.
- **Example 2. 5:**Let R be the usual metric space. Define $S,T:R\to R$ by $Sx=2x$ and $Tx=x^2$ for all $x\in R$.Then $Sx=Tx$ for $x=0,2$, but $ST0=TS0$, and $ST2\neq TS2$. S and T are occasionally weakly compatible self maps but not weakly compatible.
- **Lemma 2. 6** [3] **:** Let X be a set, f and g owc self maps of X. If f and g have a unique point of coincidence, $w=fx=gx$, then w is the unique common fixed point of f and g.

Main Results

Theorem 3.1: Let $(X, M,*)$be fuzzy metric space and let A, B, S and T be self mappings of X. Let the pairs $\{A,S\}$ and $\{B,T\}$ be *owc*. If there exist $q \in (0,1)$ s.t.

$$M(Ax,By,qt) \geq min\left\{\frac{M(Ax,Ty,t)+M(By,Sx,t)}{2}, \frac{M(Sx,Ax,t)+M(By,Ty,t)}{2}\right\} \qquad (1)$$

for all $x, y \in X$, $t > 0$ and $a,b,c,d \geq 0$ with a & b $(c \& d)$ cannot be simultaneously 0,then there exist a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover $z = w$, so that there is a unique common fixed point of A, B, S and T.

Proof: Let the pairs $\{A,S\}$ and $\{B,T\}$ be *owc*, so there are points $x, y \in X$ s.t. $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (1)

$$M(Ax,By,qt) \geq min\left\{\frac{M(Ax,Ty,t)+M(By,Sx,t)}{2}, \frac{M(Sx,Ax,t)+M(By,\ Ty,\ t)}{2}\right\}$$

$$M(Ax,By,qt) \geq min\left\{\frac{M(Ax,By,t)+M(By,Ax,t)}{2}, \frac{M(Ax,Ax,t)+M(By,By,t)}{2}\right\}$$

$$M(Ax,By,qt) \geq min\{M(Ax,By,t),1\} = M(Ax,By,t)$$

Therefore $Ax = By$ i.e. $Ax = Sx = By = Ty$. Suppose that there is another point z s.t. $Az = Sz$ then by inequality (1) we have $Az = Sz = By = Ty$ so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.2.6 w is the only common fixed point of A and S. Similarly there is a unique point $z \in X$ s.t. $z = Bz = Tz$.

Assume that $w \neq z$. We have, by inequality (1)

$$M(w,z,qt) = M(Aw,Bz,qt)$$

$$M(Aw,Bz,qt) \geq min\left\{\frac{M(Aw,Tz,t)+M(Bz,Sw,t)}{2}, \frac{M(Sw,Aw,t)+M(Bz,Tz,t)}{2}\right\}$$

$$M(w,z,qt) \geq min\left\{\frac{M(w,z,t)+M(z,w,t)}{2}, \frac{M(w,w,t)+M(z,z,t)}{2}\right\}$$

$$M(w,z,qt) \geq min\{M(w,z,t),1\} = M(w,z,t)$$

Therefore we have $z = w$, by Lemma 2.2.1 z is a common fixed point of A, B, S and T.

To prove uniqueness let u be another common fixed point of A, B, S and T. Then

$$M(z,u,qt) = M(Az,Bu,qt)$$

$$M(Az,Bu,qt) \geq min\left\{\frac{M(Az,Tu,t)+M(Bu,Sz,t)}{2}, \frac{M(Sz,Az,t)+M(Bu,\ Tu,\ t)}{2}\right\}$$

$$(z,u,qt) \geq min\left\{\frac{M(z,u,t)+M(u,z,t)}{2}, \frac{M(z,z,t)+M(u,u,t)}{2}\right\}$$

$$M(z,u,qt) \geq min\{M(z,u,t),1\} = M(z,u,t)$$

Therefore by lemma 2.2.1 we have $z = u$.

Theorem 3.2: Let $(X, M,*)$ be fuzzy metric space and let A, B, S and T be self mappings of X .Let the pairs $\{A,S\}$ and $\{B,T\}$ be *owc*. If there exist $q \in (0,1)$ s.t.

$$M(Ax,By,qt) \geq \emptyset \begin{bmatrix} \min\{(M(Ax,Ty,t)+M(By,Sx,t))/2, \\ (M(Sx,Ax,t)+M(By,Ty,t))/2\} \end{bmatrix}$$

for all $x, y \in X$, $t > 0$ and $a,b,c,d \geq 0$ with a & b $(c \& d)$ cannot be simultaneously 0, $\phi:[0,1] \to [0,1]$ such that $\phi(t) > t$ for all $0 < t < 1$, then there is a unique common fixed point of A, B, S and T.

Proof: The Proof follows from theorem 3.1.

Theorem 3.3: Let $(X,M,*)$ be fuzzy metric space and let A,B,S and T be self mappings of X .Let the pairs $\{A,S\}$ and $\{B,T\}$ be owc . If there exist $q\in(0,1)$ s.t.

$$M(Ax,By,qt) \geq \emptyset\begin{bmatrix}\{(M(Ax,Ty,t)+M(By,Sx,t))/2,\\(M(Sx,Ax,t)+M(By,Ty,t))/2\}\end{bmatrix} \ldots\text{(I)}$$

for all $x,y\in X$, $t>0$ and $a,b,c,d\geq 0$ with a & b $(c\,\&\,d)$ cannot be simultaneously $0,\phi:[0,1]^5\to[0,1]$ such that $\phi(1,t,t,t,1)>t$ for all $0<t<1$, then there is a unique common fixed point of A,B,S and T .

Proof: Let the pairs $\{A,S\}$ and $\{B,T\}$ be owc, so there are points $x,y\in X$ s.t. $Ax=Sx$ and $By=Ty$. We claim that $Ax=By$. If not, by inequality (I)

$$M(Ax,By,qt) \geq \emptyset\begin{bmatrix}\{(M(Ax,Ty,t)+M(By,Sx,t))/2,\\(M(Sx,Ax,t)+M(By,Ty,t))/2\}\end{bmatrix}$$

$$M(Ax,By,qt) \geq \emptyset\begin{bmatrix}\{(M(Ax,By,t)+M(By,Ax,t))/2,\\(M(Ax,Ax,t)+M(By,By,t))/2\}\end{bmatrix}$$

$$= \emptyset\begin{bmatrix}\{M(Ax,By,t),\\1\}\end{bmatrix} > M(Ax,By,t)$$

a contradiction, therefore $Ax=By$. i.e. $Ax=Sx=By=Ty$. Suppose that there is another point z such that $Az=Sz$ then by (3), $Az=Sz=By=Ty$, so $Ax=Az$ and $w=Ax=Tx$ is the unique point of coincidence of A and T By lemma 2.2.6 w is a unique common fixed point of A and S . Similarly there is a unique point $z\in X$ s.t. $z=Bz=Tz$.

$M(w,z,qt)=M(Aw,Bz,qt)$

$$M(Aw,Bz,qt) \geq \emptyset\begin{bmatrix}\{(M(Aw,Tz,t)+M(Bz,Sw,t))/2,\\(M(Sw,Aw,t)+M(Bz,Tz,t))/2\}\end{bmatrix}$$

$$M(w,z,qt) \geq \emptyset\begin{bmatrix}\{(M(w,z,t)+M(z,w,t))/2,\\(M(w,w,t)+M(z,z,t))/2\}\end{bmatrix}$$

$$M(w,z,qt) \geq \emptyset\begin{bmatrix}\{M(w,z,t),\\1\}\end{bmatrix} > M(w,z,t)$$

Therefore we have $z=w$, by Lemma 2.2.1. z is a common fixed point of A,B,S and T .

To prove uniqueness let u be another common fixed point of A,B,S and T . Then

$$M(z,u,qt)=M(Az,Bu,qt)\geq\emptyset\left\{\frac{M(Az,Tu,t)+M(Bu,Sz,t)}{2},\frac{M(Sz,Az,t)+M(Bu,\ Tu,\ t)}{2}\right\}$$

$$M(Az,Bu,qt)\geq\emptyset\left\{\frac{M(z,u,t)+M(u,z,t)}{2},\frac{M(z,z,t)+M(u,u,t)}{2}\right\}$$

$$M(z,u,qt)\geq\emptyset\{M(z,u,t),1\}\geq M(z,u,t)$$

Therefore by lemma 2.1 we have $z=u$.

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