Optimal Control Problem and Existence

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Abstract
With optimisation and prove the presence of the minimis, i translate the identification issue in this paper into an ideal control problem. In general, reverse problems can often be reformulated as optimization problems to minimize the functional cost, or error, based on the yield data. In request to prove the presence of the minimizewe need the following assumption on the coefficients, initial data and measurement yield data:

Assumption 3.2.1 For α > 0, assume that the coefficients a, b, c, d and the initial data φ, ϕ satisfy

$$a(x), b(x), c(x), d(x) \in C^{\alpha}(\bar{I}), \;\; m(x), n(x) \in L^2(I) \quad and \quad \phi(x), \varphi(x) \in C^{2,\alpha}(\bar{I}),$$

Where*m(x), n(x)*Comply with the homogenous Dirichlet limits Let us now define the allowable set

$$\mathcal{M} = \{a(x), c(x) \;:\; 0 < a_0 \leq a \leq a_1, \;\; 0 < c_0 \leq c \leq c_1, \;\; \nabla a, \nabla c \in L^2(I)\}$$

Also, the ideal control problem with the cost functional based on the known yield measurement at the final time like the form (2.2) as pursues: *Find (¯a(x), c¯(x)) ∈ M × M* satisfying

$$\mathcal{J}(\bar{a}, \bar{c}) = \min_{a,c \in \mathcal{M}} \mathcal{J}(a, c), \quad (2.3)$$

Where

$$\mathcal{J}(a,c) = \frac{1}{2}\int_I \left(|u(x,T;a) - m(x)|^2 + |v(x,T;c) - n(x)|^2\right) \mathrm{d}x + \frac{\wp}{2}\int_I \left(|\nabla a|^2 + |\nabla c|^2\right) \mathrm{d}x \quad (2.4)$$

And (u, V) is the system solution (3.1.1) with the a(x), c(x), and M coefficients. The constants a0, a1 and c0, c1 are indicated and §fix is the parameter for regularization.

Theorem 3.2.1 Let 0 < α < 1 and the coefficients a(x), b(x), c(x), d(x) ∈ C α (I). Then the system (2.1.1) has a unique solution $u(x,t), v(x,t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega}_T)$.

This theorem demonstrates the existence of an optimal control (a(x), c(x)) (along with M) to minimize cost-functional J (a, c).

Theorem 3.2.2 Suppose (u, v) is the solution of the system (2.2). Then there exists a minimize (a(x), c(x)) ∈ M × M such that

$$\mathcal{J}(\bar{a}, \bar{c}) = \min_{a,c \in \mathcal{M}} \mathcal{J}(a, c).$$

Proof. Note from J (a, c) the functional J (a, c) being non-negative and therefore the lower bound being the largest. The minimizing sequence is let (un, vn, an, cn), for example,

$$\inf_{a,c \in \mathcal{M}} \mathcal{J}(a, c) \le \mathcal{J}(a_n, c_n) \le \inf_{a,c \in \mathcal{M}} \mathcal{J}(a, c) + \frac{1}{n}, \quad \text{for} \quad n = 1, 2, \cdots.$$

Taking $J(a_n, c_n) \le C$ into account, we easily deduce that

$$\|\nabla a_n\|_{L^2(I)} + \|\nabla c_n\|_{L^2(I)} \le C,$$

Where the constant C is independent of n Then the Sobolev imbedding

$H^1(I) \subset C^{\alpha}(\bar{I})$ for $0 < \alpha \le \frac{1}{2}$lead to

$$H^1(I) \subset C^{\alpha}(\bar{I}) \text{ for } 0 < \alpha \le \tfrac{1}{2} \text{ lead to}$$

$$\|a_n\|_{C^{\frac{1}{2}}(I)} + \|c_n\|_{C^{\frac{1}{2}}(I)} \le C.$$

Thus, classical solutions for parabolic equations exist, we have

$$\|u_n\|_{C^{\frac{1}{2}, \frac{1}{4}}(\bar{\Omega}_T)} + \|v_n\|_{C^{\frac{1}{2}, \frac{1}{4}}(\bar{\Omega}_T)} \le C,$$

and for any ωT b ΩT ; we also get

$$\|u_n\|_{C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\omega_T)} + \|v_n\|_{C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\omega_T)} \le C.$$

Then there exists a subsequence of (u_n, v_n, a_n, c_n), again denoted by (u_n, v_n, a_n, c_n), such that

$$\begin{aligned}(a_n, c_n) &\rightarrow (\bar{a}, \bar{c}) \in C^{\frac{1}{2}}(I) \times C^{\frac{1}{2}}(I) \text{ uniformly on } C^{\alpha}(\bar{I}) \times C^{\alpha}(\bar{I}),\\ (u_n, v_n) &\rightarrow (\phi, \varphi) \text{ uniformly on } (C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T) \cap C^{2+\alpha,\ 1+\frac{\alpha}{2}}_{\text{loc}}(\Omega_T))^2.\end{aligned}$$

Hence replacing (u, v, a, c) in (2.1.1) by (u_n, v_n, a_n, c_n) and passing to the limit, one sees that $(\phi, \varphi, \bar{a}, \bar{c})$ satisfies the system (2.1.1). In addition, we achieve the weak semi-continuity of the L 2 -norm with the Lebesgue control convergence theorem

$$\mathcal{J}(\bar{a}, \bar{c}) \leq \liminf_{n\to\infty} \mathcal{J}(a_n, c_n) = \min_{a,c\in\mathcal{M}} \mathcal{J}(a, c),$$

Whence

$$\mathcal{J}(\bar{a}, \bar{c}) = \min_{a,c\in\mathcal{M}} \mathcal{J}(a, c).$$

Thus $(\bar{a}, \bar{c}) := (a, c)$ is an optimal solution of the optimal control problem (2.2.1)-(2.2.3). Hence the proof is complete.

Optimality Conditions and Stability

In this section, we have established as a reversal problem the essential optimal condition for ideal control of the two smooth coefficients (a(x) and c(x) defined in the previous section and estimate the stability of this problem. The main theorem estimates the error between two top-limit materials, which are given by certain Sobolev solutions standards at t = T in a(x) and ea(x) (x) An important optimum conditions are the main ingredient in proof of such a stability estimate. We now have essential optimal conditions that must be fulfilled by every ideal control (a, c). Assume (p, q) the deputy system solution (2.1.1) is the form.

$$\left.\begin{aligned}&-p_t - p_{xx} + ap + dq = 0,\ (x,t) \in \Omega_T,\\ &-q_t - q_{xx} + cq + bp = 0,\ (x,t) \in \Omega_T,\\ &p(x,T) = u(x,T) - m(x),\ x \in I,\\ &q(x,T) = v(x,T) - n(x),\ x \in I,\\ &p(0,t) = p(1,t) = q(0,t) = q(1,t) = 0,\ t \in [0,T),\end{aligned}\right\} \quad (2.5)$$

Where m, n are the values of the solutions of the system (2.1.1) at final time t = T.

Theorem 3.3.1 Let (a, c) be the solution of the optimal control problem. Then there exists a lot of functions (u, v, p, q; a, c) satisfying the following optimal condition

$$\int_{\Omega_T} (pu(a-k) + qv(c-l))\,\mathrm{d}x\mathrm{d}t + N\int_I [\nabla a \cdot \nabla(k-a) + \nabla c \cdot \nabla(l-c)]\,\mathrm{d}x \geq 0,$$

for any k, l ∈ M

Proof. For any k, l ∈ M and 0 ≤ δ ≤ 1, set

$$a_\delta = (1-\delta)a + \delta k \in \mathcal{M} \quad \text{and} \quad c_\delta = (1-\delta)c + \delta l \in \mathcal{M}.$$

Then there exists a solution (uδ, vδ) of the system with the coefficients a = aδ and c = cδ satisfying

$$\mathcal{J}_\delta = \mathcal{J}(a_\delta, c_\delta) = \frac{1}{2}\int_I [|u_\delta - m(x)|^2 + |v_\delta - n(x)|^2]\,\mathrm{d}x + \frac{\wp}{2}\int_I (|\nabla a_\delta|^2 + |\nabla c_\delta|^2)\,\mathrm{d}x,$$

Where $u_\delta = u(x,T;a_\delta)$ and $v_\delta = v(x,T;c_\delta)$. now taking the Frechet derivative of $\mathcal{J}_\delta$, we have

$$\begin{aligned}\left.\frac{\mathrm{d}\mathcal{J}_\delta}{\mathrm{d}\delta}\right|_{\delta=0} &= \int_I \left([u_\delta - m(x)] \left.\frac{\partial u_\delta}{\partial \delta}\right|_{\delta=0} + [v_\delta - n(x)] \left.\frac{\partial v_\delta}{\partial \delta}\right|_{\delta=0} \right) \mathrm{d}x \\ &\quad + \wp \int_I [\nabla a \cdot \nabla(k-a) + \nabla c \cdot \nabla(l-c)]\,\mathrm{d}x.\end{aligned}$$

Moreover (a, c) is the optimal solution and therefore

$$\left.\frac{\mathrm{d}\mathcal{J}_\delta}{\mathrm{d}\delta}\right|_{\delta=0} \geq 0. \quad (2.6)$$

If we take $(\bar{u}_\delta, \bar{v}_\delta) = (\frac{\partial u_\delta}{\partial \delta}, \frac{\partial v_\delta}{\partial \delta})$, then $(\bar{u}_\delta, \bar{v}_\delta)$ satisfies the following system with the coefficients $(a_\delta, b, c_\delta, d)$,

$$\left.\begin{array}{l}(\bar{u}_\delta)_t - (\bar{u}_\delta)_{xx} + a_\delta \bar{u}_\delta + (k-a)u_\delta + b\bar{v}_\delta = 0, \ (x,t) \in \Omega_T, \\ (\bar{v}_\delta)_t - (\bar{v}_\delta)_{xx} + c_\delta \bar{v}_\delta + (l-c)v_\delta + d\bar{u}_\delta = 0, \ (x,t) \in \Omega_T, \\ \bar{u}_\delta(x,0) = \bar{v}_\delta(x,0) = 0, \ x \in I, \\ \bar{u}_\delta(0,t) = \bar{u}_\delta(1,t) = \bar{v}_\delta(0,t) = \bar{v}_\delta(1,t) = 0, \ t \in (0,T].\end{array}\right\}$$

Taking $\xi = \bar{u}_\delta|_{\delta=0}$ and $\eta = \bar{v}_\delta|_{\delta=0}$, we see that (ξ, η) satisfies the following system

$$\left.\begin{array}{l} \xi_t - \xi_{xx} + a\xi + b\eta = (a-k)u, \ (x,t) \in \Omega_T, \\ \eta_t - \eta_{xx} + c\eta + d\xi = (c-l)v, \ (x,t) \in \Omega_T, \\ \xi(x,0) = \eta(x,0) = 0, \ x \in I, \\ \xi(0,t) = \xi(1,t) = \eta(0,t) = \eta(1,t) = 0, \ t \in (0,T], \end{array}\right\}$$

Where $u_\delta|_{\delta=0} = u$ and $v_\delta|_{\delta=0} = v$. Now, from (2.3.2), we have

$$\int_I ([u(x,T;a) - m(x)]\xi(x,T) + [v(x,T;c) - n(x)]\eta(x,T))\,\mathrm{d}x \\ + \wp \int_I [\nabla a \cdot \nabla(k-a) + \nabla c \cdot \nabla(l-c)]\,\mathrm{d}x \geq 0.$$

Further from (2.3.1), the last inequality reads as

$$\int_I (p(x,T)\xi(x,T) + q(x,T)\eta(x,T))\,\mathrm{d}x \\ + \wp \int_I [\nabla a \cdot \nabla(k-a) + \nabla c \cdot \nabla(l-c)]\,\mathrm{d}x \geq 0. \quad (2.6)$$

Suppose (p, q) is the solution of the system (2.3.1). Multiplying the first equation of (2.3.1) by ξ and using Green's theorem, we have

$$0 = \int_{\Omega_T} \xi(-p_t - p_{xx} + ap + dq)\,\mathrm{d}x\mathrm{d}t$$

$$= -\int_I \xi p\big|_0^T \mathrm{d}x + \int_{\Omega_T} p\,(\xi_t - \xi_{xx} + a\xi + b\eta - b\eta)\,\mathrm{d}x\mathrm{d}t + \int_{\Omega_T} dq\xi\,\mathrm{d}x\mathrm{d}t \\ = -\int_I \xi(x,T)p(x,T)\,\mathrm{d}x + \int_{\Omega_T} p\,(\xi_t - \xi_{xx} + a\xi + b\eta)\,\mathrm{d}x\mathrm{d}t + \int_{\Omega_T} (dq\xi - bp\eta)\,\mathrm{d}x\mathrm{d}t.$$

From the system (3.3.3), one gets

$$\int_I \xi(x,T)p(x,T)\,\mathrm{d}x = \int_{\Omega_T} pu(a-k)\,\mathrm{d}x\mathrm{d}t + \int_{\Omega_T} (dq\xi - bp\eta)\,\mathrm{d}x\mathrm{d}t. \quad (2.7)$$

Similarly, from the second equation of (3.3.1) and (3.3.3), we have

$$\int_I \eta(x,T)q(x,T)\,\mathrm{d}x = \int_{\Omega_T} qv(c-l)\,\mathrm{d}x\mathrm{d}t + \int_{\Omega_T} (bp\eta - dq\xi)\,\mathrm{d}x\mathrm{d}t. \quad (2.8)$$

Substituting the values of (3.3.5) and (3.3.6) in (3.3.4), one easily completes the proof of Theorem 3.3.1.

Basic Lemmas

In request to prove the main stability estimates, we need the vitality estimates of the system (3.1.1) and its promotion joint system. Assume (u, e ve) is the solution of the following system

$$\left.\begin{array}{l} \widetilde{u}_t - \widetilde{u}_{xx} + \widetilde{a}(x)\widetilde{u} + b(x)\widetilde{v} = 0,\ (x,t)\in\Omega_T, \\ \widetilde{v}_t - \widetilde{v}_{xx} + \widetilde{c}(x)\widetilde{v} + d(x)\widetilde{u} = 0,\ (x,t)\in\Omega_T, \\ \widetilde{u}(x,0) = \phi(x),\ \widetilde{v}(x,0) = \varphi(x),\ x\in I, \\ \widetilde{u}(0,t) = \widetilde{u}(1,t) = \widetilde{v}(0,t) = \widetilde{v}(1,t) = 0,\ t\in(0,T]. \end{array}\right\} \quad (2.9)$$

Set $U = u - \widetilde{u},\ V = v - \widetilde{v},\ \mathcal{A} = a - \widetilde{a}$ and $\mathcal{C} = c - \widetilde{c}$ so that the subtraction of (3.3.7) from (2.1.1) yields

$$\left.\begin{array}{l} U_t - U_{xx} + aU + bV = -\mathcal{A}\widetilde{u},\ (x,t)\in\Omega_T, \\ V_t - V_{xx} + cV + dU = -\mathcal{C}\widetilde{v},\ (x,t)\in\Omega_T, \\ U(x,0) = 0,\ V(x,0) = 0,\ x\in I, \\ U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0,\ t\in(0,T]. \end{array}\right\} \quad (2.10)$$

In proving the principal outcome of this section the following lemmas play the most important role.

Lemma 3.3.1. Let (U, V) be the solution of the system (2.3.8). Then we have the following estimate:

$$\max_{0\le t\le T}\int_I (|U|^2 + |V|^2)\ \mathrm{d}x \le \exp(MT)\left(\max_{x\in I}|\mathcal{A}|^2\int_{\Omega_T}|\widetilde{u}|^2\,\mathrm{d}x\mathrm{d}t + \max_{x\in I}|\mathcal{C}|^2\int_{\Omega_T}|\widetilde{v}|^2\,\mathrm{d}x\mathrm{d}t\right),$$

Where the constant $M = (2 + \max_{x\in I}|b|^2 + \max_{x\in I}|d|^2)$.

Proof. Multiply the first equation of (2.3.8) by U and integrate over I to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2_{L^2(I)} + \int_I |U_x|^2\,\mathrm{d}x + \int_I a|U|^2\,\mathrm{d}x = -\int_I bUV\,\mathrm{d}x - \int_I \mathcal{A}U\widetilde{u}\,\mathrm{d}x.$$

Using the assumption on the coefficient a and applying Cauchy's inequality, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2_{L^2(I)} + \int_I |U_x|^2\,\mathrm{d}x + a_0\int_I |U|^2\,\mathrm{d}x$$

$$\leq \int_I |U|^2\,\mathrm{d}x + \frac{1}{2}\max_{x\in I}|\mathcal{A}|^2\int_I |\widetilde{u}|^2\,\mathrm{d}x + \frac{1}{2}\max_{x\in I}|b|^2\int_I |V|^2\,\mathrm{d}x.$$

Similarly, from the second equation of (3.3.8) and the assumption on c, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V\|^2_{L^2(I)} + \int_I |V_x|^2\,\mathrm{d}x + c_0\int_I |V|^2\,\mathrm{d}x$$

$$\leq \int_I |V|^2\,\mathrm{d}x + \frac{1}{2}\max_{x\in I}|\mathcal{C}|^2\int_I |\widetilde{v}|^2\,\mathrm{d}x + \frac{1}{2}\max_{x\in I}|d|^2\int_I |U|^2\,\mathrm{d}x.$$

Now coupling the last two estimates, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|U\|^2_{L^2(I)} + \|V\|^2_{L^2(I)}\right] \;\leq\; M(\|U\|^2_{L^2(I)} + \|V\|^2_{L^2(I)})$$

Simultaneous Identification of Parameters and Initial Data of Reaction Diffusion System

In the last decades, much attention has been paid to the work of spatial impacts in biodiversity conservation in literature. Many scientists have formulated and examined spatial population models and, in particular, reaction diffusion equations. Reaction diffusion models are spatial and incorporate different natural amounts regularly, such as location or time-specific parameters.

But quantifying or figuring the model parameters with the vital accuracy, especially in living life forms, cannot be constantly imagined. In these cases, it is important to use mathematical methods to estimate model parameters. In complex systems with numerous parameters, parameter estimation is also a critical way to detect repetitive parameters (and henceforth key parameters). In the following system of reaction diffusion we obtain two spatially subordinate reaction parameters and the initial data that are often applied for several physical applications, such as: in scientific and medical sciences, increasing and developing malignancy in the environment and angiogenesis, prey predatory and insect dispersal system in science, reaction inside sig:

$$\left.\begin{array}{ll} u_t - \Delta u + a(x)u + b(x)v = 0, & (x,t) \in \Omega_T = \Omega \times (0,T], \\ v_t - \Delta v + c(x)v + d(x)u = 0, & (x,t) \in \Omega_T, \\ u(x,0) = \phi(x),\ v(x,0) = \varphi(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & (x,t) \in \Sigma = \partial\Omega \times (0,T], \end{array}\right\}$$

The domain > 0 is an arbitrary but fixed time, and the concentrations of both agents are indicated by u(x, t) and v(x, t). Unknown initial conditions α(x) and ć(x), depending only upon x, are regular enough and sufficient smoothness is presumed for the unknown coeffcients a(x), c(x) and coefficients d(x), b(x), d(x) and time t is maintained. Suppose we assume that additional temperature can be provided for reverse thermal problems, for example by arbitrarily fixing additional measurements; sub domain ω ⊂ Ω,

$$u(x,t) = \pi(x,t) \quad v(x,t) = \varrho(x,t), \quad (x,t) \in \omega \times (0,T),$$

And the additional temperatures u(x, t), v(x, t) are given at some final time t = T for all x ∈ Ω, that is

$$u(x,T) = m(x), \quad v(x,T) = n(x), \quad x \in \Omega,$$

Where α(x, t),(x, t), m(x) and n(x) functions are known, m(x) and n(x) also meet the same bounds of Dirichlet. The objective of this chapter is to obtain stability forecasts simultaneously in two equation reaction diffusion system for the solution of a system on an arbitrary subdomaines, in the two reaction diffusion equations, a(x), c(x) and initial data ̈(x), ̈(x). In achieving the stability estimate in the reaction diffusion system the optimum control framework plays the critical role.

Different methods based on various observations have been used to discuss the reconstruction of paramètres, initial or boundary data in the literature. In the literature, a few publications are available to study the identification of multiple parameters. The multipara meter synchronization of a single parabola equation has been examined. They examine the concurrent rehabilitation of the initial temperature and warmth radiogenic coefficient in the warmth-driving system and develop unique stability in the solution to the reverse problem by regularizing the reconstruction procedure of Tikhonov in accordance with the L 2 inclination standard. By creating a suitable cost function, the inverse problem is converted into an optimal control problem. Since the optimization plot is one of the most crucial methods for dealing with the numerical reconstruction, its position is generally investigated and its presence is demonstrated. In order to optimize the inversion problem in the reconstruction of the source conditions by using powerless solutions in the linear parabola differential equation, the uniqueness and stability of minimizers have been developed later. Paragraph 2 shows that, by simultaneously reconstructing spatial watch coefficients, the stability and a local uniqueness of a linear diffusion were achieved from the final measuring data. This work results in stability for the recovery of two times independent coefficients x and c(x), and shows the initial temperatures in linear diffusion of reactions. Furthermore, the strategy we have adopted here enables us to achieve stability with some halfway information on the Sobolev solution at the top threshold and it should be stressed that the impact on reaction dissemination systems is not such in literature.

Optimal Control Problem

In this section, we turn the problem in identification into an optimum control problem through optimization theory and prove that the minimizer exists. The reworded problem of optimization involves a fee-based observation of the self-assertive fixed sub-domain and the final time of data. This sort of cost functional has been considered for the general parabolic equation. We stretch out the concept to the system of parabolic equations. All through this section, the measurement functions fulfill the following assumption.

Assumption 3.2.1 The arbitrary fixed sub domain observation $\pi(x,t),\ \varrho(x,t) \in$

$L^2(0,T;L^2(\omega))$ And the final time over specified measurement $m(x), n(x) \in L^2(\Omega)$,

Where m(x), n(x) fulfills the Dirichlet homogeneous limit conditions, the allowable set for the reaction coefficient and initial data are now defined by

$$\left.\begin{array}{l}\mathcal{M}_1 = \{a(x), c(x)\ :\ 0 < a_0 \le a \le a_1,\ \ 0 < c_0 \le c \le c_1,\ a, c \in H^1(\Omega)\},\\ \mathcal{M}_2 = \{\phi(x), \varphi(x) :\ 0 < \phi \le \hat{\phi},\ \ 0 < \varphi \le \hat{\varphi},\ \ \ \phi, \varphi \in H^1(\Omega)\},\end{array}\right\} \quad (3.1)$$

and the optimal control problem with the cost functional based on the known output measurement at the arbitrary fixed sub domain in the form (3.1.2) and the final time in the form (3.1.3) is stated as follows:

Find $(\bar{a}(x), \bar{c}(x), \bar{\phi}(x), \bar{\varphi}(x)) \in \mathcal{M}_1^2 \times \mathcal{M}_2^2$ satisfying

$$\mathcal{J}(\bar{a}, \bar{c}, \bar{\phi}, \bar{\varphi}) = \min_{(a,c,\phi,\varphi)\in\mathcal{M}_1^2\times\mathcal{M}_2^2} \mathcal{J}(a, c, \phi, \varphi), \quad (3.2)$$

Where

$$\begin{aligned}\mathcal{J}(a, c, \phi, \varphi) =& \frac{1}{2}\iint_{\omega_T} \left[|u(a,\phi) - \pi(x,t)|^2 + |v(c,\varphi) - \varrho(x,t)|^2\right] \mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{2}\iint_{\Omega_\sigma} \left(|u(a,\phi) - m(x)|^2 + |v(c,\varphi) - n(x)|^2\right) \mathrm{d}x\mathrm{d}t \\ &+ \frac{\wp_1}{2}\int_\Omega \left(|\nabla a|^2 + |\nabla c|^2\right) \mathrm{d}x + \frac{\wp_2}{2}\int_\Omega \left(|\nabla \phi|^2 + |\nabla \varphi|^2\right) \mathrm{d}x,\end{aligned} \quad (3.3)$$

and $\omega_T = \omega \times [0,T]$ and $\Omega_\sigma = \Omega \times [T-\sigma, T]$, $(u(a,\phi), v(c,\varphi)) := (u(x,t;a,\phi),$ $v(x,t;c,\varphi))$ is the solution of the system (3.1.1) for the given coefficients $a(x), c(x) \in \mathcal{M}_1$ and the given initial data $\phi(x), \varphi(x) \in \mathcal{M}_2$. The constants a_0, a_1, c_0, c_1 and $\hat{\phi}, \hat{\varphi}$ are known and $\wp_1, \wp_2$ are the regularization parameters.

Lemma 3.2.1 Let (u, v) be the solution of the system (3.1.1). Then we have

$$\max_{0\leq t\leq T}\int_{\Omega}(|u|^2+|v|^2)\,\mathrm{d}x\leq\exp(MT)\big(\|\phi\|^2_{L^2(\Omega)}+\|\varphi\|^2_{L^2(\Omega)}\big),$$

$$where\ M=\left(2+\max_{x\in\Omega}|b|^2+\max_{x\in\Omega}|d|^2\right)$$

Proof. Multiplying the first and second equation of (3.1.1) respectively by u and v and integrating on **Ω**, we

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\right]+\int_{\Omega}\left(|\nabla u|^2+|\nabla v|^2\right)\,\mathrm{d}x+\int_{\Omega}\left(a|u|^2+c|v|^2\right)\,\mathrm{d}x$$
$$\leq\frac{1}{2}\left(2+\max_{x\in\Omega}|b|^2+\max_{x\in\Omega}|d|^2\right)\left(\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\right).$$

From the assumptions on a and c, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\right]\leq\frac{M}{2}\left(\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\right),$$

Whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(-Mt)\left(\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\right)\right]\leq 0.$$

The integration on (0, t) thus concludes the evidence. With regard to the Lemma 3.2.1 and the measurement assumptions, it is easy to understand that the cost function for each (a, c, β, μ). The inverse problem is currently transformed into a problem of optimization control. We have to show that for the minimization problem something like one exists (3.2.1)- (3.2.3). To prove the presence of minimize, we need the following Lemma:

Lemma 3.2.2 For any sequences $\{a_n,c_n,\phi_n,\varphi_n\}\ in\ \mathcal{M}_1^2\times\mathcal{M}_2^2$ which converge to some $\{a,c,\phi,\varphi\}\ in\ \mathcal{M}_1^2\times\mathcal{M}_2^2\ as\ n\to\infty,$

$$\lim_{n\to\infty}\left(\iint_{\omega_T}|u(a_n,\phi_n)-\pi(x,t)|^2\,\mathrm{d}x\mathrm{d}t+\iint_{\omega_T}|v(c_n,\varphi_n)-\varrho(x,t)|^2\,\mathrm{d}x\mathrm{d}t\right)$$

$$=\iint_{\omega_T}|u(a,\phi)-\pi(x,t)|^2\,\mathrm{d}x\mathrm{d}t+\iint_{\omega_T}|v(c,\varphi)-\varrho(x,t)|^2\,\mathrm{d}x\mathrm{d}t,$$

$$\lim_{n\to\infty}\left(\iint_{\Omega_\sigma}|u(a_n,\phi_n)-m(x)|^2\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_\sigma}|v(c_n,\varphi_n)-n(x)|^2\,\mathrm{d}x\mathrm{d}t\right)$$
$$=\iint_{\Omega_\sigma}|u(a,\phi)-m(x)|^2\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_\sigma}|v(c,\varphi)-n(x)|^2\,\mathrm{d}x\mathrm{d}t.$$

Proof. For the choice of the parameters $(a,c,\phi,\varphi)\in\mathcal{M}_1^2\times\mathcal{M}_2^2$, multiply the first and second equation of (3.1.1) by ψ and ψb respectively and combine them to get

$$\int_\Omega\left(u_t(a,\phi)\psi+v_t(c,\varphi)\widehat{\psi}\right)\,\mathrm{d}x+\int_\Omega\left(\nabla u(a,\phi)\cdot\nabla\psi+\nabla v(c,\varphi)\cdot\nabla\widehat{\psi}\right)\,\mathrm{d}x$$

$$+\int_\Omega\left(a(x)u(a,\phi)\psi+c(x)v(c,\varphi)\widehat{\psi}\right)\,\mathrm{d}x$$

$$+\int_\Omega\left(b(x)v(c,\varphi)\psi+d(x)u(a,\phi)\widehat{\psi}\right)\,\mathrm{d}x=0, \quad (3.4)$$

where $\psi,\widehat{\psi}$ are the test functions in $H_0^1(\Omega)$. By taking $\psi=u(a,\phi)$, $\widehat{\psi}=v(c,\varphi)$

and integrating with respect to t, we get

$$\int_\Omega\left(|u(a,\phi)|^2+|v(c,\varphi)|^2\right)\,\mathrm{d}x+\iint_{\Omega_T}\left(|\nabla u(a,\phi)|^2+|\nabla v(c,\varphi)|^2\right)\,\mathrm{d}x\mathrm{d}t$$
$$\leq C\iint_{\Omega_T}\left(|u(a,\phi)|^2+|v(c,\varphi)|^2\right)\,\mathrm{d}x\mathrm{d}t+\int_\Omega\left(|\phi|^2+|\varphi|^2\right)\,\mathrm{d}x, \quad (3.5)$$

Any t [0, t] for every t [0]. The above equation and the assumption on the allowable set imply that $\{u(a_n,\phi_n),v(c_n,\varphi_n)\}$ are bounded in the space $L^2(0,T;H_0^1(\Omega))^2$. . There exists a subsequence, still denoted by $\{u(a_n,\phi_n),v(c_n,\varphi_n)\}$, such

$$\left.\begin{aligned}&u(a_n,\phi_n)\rightharpoonup\bar{u}\text{ weakly in }L^2(0,T;H_0^1(\Omega)),\\&v(c_n,\varphi_n)\rightharpoonup\bar{v}\text{ weakly in }L^2(0,T;H_0^1(\Omega)).\end{aligned}\right\}\quad(3.6)$$

Next we have to prove that the converging point is $(\bar{u}, \bar{v}) = (u(a,\phi), v(c,\varphi))$.

In order to prove that, choose a function $\chi(t) \in C^1[0,T]$ with the property that χ(T) = 0.
Multiply (3.2.4) by $\chi(t)$ for $\{u(a_n, \phi_n), v(c_n, \phi_n)\}$ and integrate over t to

$$\iint_{\Omega_T}\Big[\Big(\nabla u(a_n,\phi_n)\cdot\nabla\psi+\nabla v(c_n,\varphi_n)\cdot\nabla\widehat{\psi}\Big)+\Big(u_t(a_n,\phi_n)\psi+v_t(c_n,\varphi_n)\widehat{\psi}\Big)\Big]\,\chi\,\mathrm{d}x\mathrm{d}t$$

$$+\iint_{\Omega_T}\Big[\Big(a_n u(a_n,\phi_n)\psi+c_n v(c_n,\varphi_n)\widehat{\psi}\Big) + \Big(bv(c_n,\varphi_n)\psi+du(a_n,\phi_n)\widehat{\psi}\Big)\Big]\,\chi\,\mathrm{d}x\mathrm{d}t=0. \quad (3.7)$$

After some manipulation along with (3.2.5)-(3.2.6), (3.2.7) can be rewrite

$$\iint_{\Omega_T}\Big(\nabla\bar{u}\cdot\nabla\psi+\nabla\bar{v}\cdot\nabla\widehat{\psi}\Big)\,\chi(t)\,\mathrm{d}x\mathrm{d}t+\int_\Omega\Big[\phi\psi(0)+\varphi\widehat{\psi}(0)\Big]\,\chi(0)\,\mathrm{d}x$$
$$-\iint_{\Omega_T}\Big(\bar{u}\psi+\bar{v}\widehat{\psi}\Big)\,\chi_t\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}\Big(a\bar{u}\psi+c\bar{v}\widehat{\psi}+b\bar{v}\psi+d\bar{u}\widehat{\psi}\Big)\,\chi(t)\,\mathrm{d}x\mathrm{d}t=0, \quad (3.8)$$

which is valid for any $\chi(t)\in[0,T]$ with $\chi(T)=0$. Hence we

$$\iint_{\Omega_T}\Big(\nabla\bar{u}\cdot\nabla\psi+\nabla\bar{v}\cdot\nabla\widehat{\psi}\Big)\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}\Big(\bar{u}_t\psi+\bar{v}_t\widehat{\psi}\Big)\,\mathrm{d}x\mathrm{d}t$$
$$+\iint_{\Omega_T}\Big(a\bar{u}\psi+c\bar{v}\widehat{\psi}+b\bar{v}\psi+d\bar{u}\widehat{\psi}\Big)\,\mathrm{d}x\mathrm{d}t=0,\quad\forall\psi,\widehat{\psi}\in H_0^1(\Omega), \quad (3.9)$$

and $\bar{u}(0)=\phi, \bar{v}(0)=\varphi.$. Therefore, by the definition of

$u(a,\phi)$ and $v(c,\varphi)$, we get u¯ = u(a, φ), v¯ = v(c, ϕ). Now we are ready to prove the lemma. ma. The equation (3.2.4) can be rewritten as follows

$$\int_\Omega\Big((u_n-m)_t\psi+(v_n-n)_t\widehat{\psi}\Big)\,\mathrm{d}x+\int_\Omega\Big(\nabla(u_n-m)\cdot\nabla\psi+\nabla(v_n-n)\cdot\nabla\widehat{\psi}\Big)\,\mathrm{d}x$$

$$+\int_\Omega\Big(a_n(u_n-m)\psi+c_n(v_n-n)\widehat{\psi}+b(v_n-n)\psi+d(u_n-m)\widehat{\psi}\Big)\,\mathrm{d}x$$
$$=\int_\Omega\Big(\nabla m\cdot\nabla\psi+\nabla n\cdot\nabla\widehat{\psi}\Big)\,\mathrm{d}x-\int_\Omega\Big(a_n m\psi+c_n n\widehat{\psi}+bn\psi+dm\widehat{\psi}\Big)\,\mathrm{d}x, \quad (3.10)$$

where $u_n = u(a_n, \phi_n),\ v_n = v(c_n, \varphi_n),\ m = m(x)$ and $n = n(x)$. Further, taking $(\psi, \widehat{\psi}) = (u_n - m, v_n - n)$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(|u_n - m|^2 + |v_n - n|^2\right) \mathrm{d}x + \int_\Omega \left(|\nabla(u_n - m)|^2 + |\nabla(v_n - n)|^2\right) \mathrm{d}x$$
$$+ \int_\Omega \left(a_n|u_n - m|^2 + c_n|v_n - n|^2\right) \mathrm{d}x + \int_\Omega (b+d)(v_n - n)(u_n - m)\,\mathrm{d}x$$

$$= -\int_\Omega (bn(u_n - m) + dm(v_n - n))\,\mathrm{d}x - \int_\Omega (a_n m(u_n - m) + c_n n(v_n - n))\,\mathrm{d}x$$

$$+ \int_\Omega (\nabla m \cdot \nabla(u_n - m) + \nabla n \cdot \nabla(v_n - n))\,\mathrm{d}x. \quad (3.11)$$

Similar relations hold for u and v for the choice $(\psi, \widehat{\psi}) = (u - m, v - n)$, (3.12)

Where $u = u(a, \phi),\ v = v(c, \varphi)$. Subtracting (3.2.12) from (3.2.11) and after some manipulate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_n - u\|^2_{L^2(\Omega)} + \|v_n - v\|^2_{L^2(\Omega)}\right) + \left(\|\nabla(u_n - u)\|^2_{L^2(\Omega)} + \|\nabla(v_n - v)\|^2_{L^2(\Omega)}\right)$$

$$+ \int_\Omega \left(a_n(x)|u_n - u|^2 + c_n(x)|v_n - v|^2\right) \mathrm{d}x = \mathcal{K}_1, \quad (3.13)$$

Where

$$\mathcal{K}_1 = \int_\Omega [u(a - a_n)(u_n - u) + v(c - c_n)(v_n - v)]\,\mathrm{d}x + \int_\Omega (b+d)\,(u - u_n)(v_n - v)\,\mathrm{d}x$$
$$+ \int_\Omega \left[(a_n - a)|u - m|^2 + (c_n - c)|v - n|^2\right] \mathrm{d}x.$$

Integrating over (0, t), for any t ≤ T, we get

$$\|u_n - u\|^2_{L^2(\Omega)} + \|v_n - v\|^2_{L^2(\Omega)} \le 2\int_0^T \mathcal{K}_1 \mathrm{d}t + \|\phi_n - \phi\|^2_{L^2(\Omega)} + \|\varphi_n - \varphi\|^2_{L^2(\Omega)}.$$

By the weak convergence of $\{u(a_n, \phi_n), v(c_n, \varphi_n)\}$ and the assumed convergence on $\{a_n, c_n, \phi_n, \varphi_n\}$, it is easy to show that

$$\int_0^T \mathcal{K}_1 \mathrm{d}t + \|\phi_n - \phi\|^2_{L^2(\Omega)} + \|\varphi_n - \varphi\|^2_{L^2(\Omega)} \to 0 \quad as \quad n \to \infty.$$

Thus we have

$$\max_{0\leq t\leq T}\int_\Omega (|u|^2+|v|^2)\,\mathrm{d}x \leq \exp(MT)\big(\|\phi\|^2_{L^2(\Omega)}+\|\varphi\|^2_{L^2(\Omega)}\big),$$

$$where\ M=\left(2+\max_{x\in\Omega}|b|^2+\max_{x\in\Omega}|d|^2\right).$$

Proof Multiplying the first and second equation of (3.1.1) respectively by u and v and integrating on **Ω**, we

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\right]+\int_\Omega \left(|\nabla u|^2+|\nabla v|^2\right)\,\mathrm{d}x+\int_\Omega \left(a|u|^2+c|v|^2\right)\,\mathrm{d}x$$
$$\leq \frac{1}{2}\left(2+\max_{x\in\Omega}|b|^2+\max_{x\in\Omega}|d|^2\right)\left(\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\right).$$

From the assumptions on a and c, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\right]\leq\frac{M}{2}\left(\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\right),$$

Whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(-Mt)\left(\|u\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\right)\right]\leq 0.$$

Thus the integration upon (0, t) concludes the proof.

From Lemma 3.2.1 and we can easily understand that the cost function is well defined for each one of the measurements assumptions (a, c, ć, ±). The inverse problem is currently transformed into a problem of optimization control. We must demonstrate that the minimization problem (3.2.1) exists at no less than one minimum (3.2.3). We need the following to demonstrate the presence of minimization. Lemma:

Lemma 3.2.2. For any sequences $\{a_n, c_n, \phi_n, \varphi_n\}$ *in* $\mathcal{M}_1^2\times\mathcal{M}_2^2$ which converge to *some* $\{a,c,\phi,\varphi\}$ *in* $\mathcal{M}_1^2\times\mathcal{M}_2^2$ *as* $n\to\infty$, *we have*

$$\lim_{n\to\infty}\left(\iint_{\omega_T}|u(a_n,\phi_n)-\pi(x,t)|^2\,\mathrm{d}x\mathrm{d}t+\iint_{\omega_T}|v(c_n,\varphi_n)-\varrho(x,t)|^2\,\mathrm{d}x\mathrm{d}t\right)$$
$$=\iint_{\omega_T}|u(a,\phi)-\pi(x,t)|^2\,\mathrm{d}x\mathrm{d}t+\iint_{\omega_T}|v(c,\varphi)-\varrho(x,t)|^2\,\mathrm{d}x\mathrm{d}t,$$

and

$$\lim_{n\to\infty}\left(\iint_{\Omega_\sigma}|u(a_n,\phi_n)-m(x)|^2\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_\sigma}|v(c_n,\varphi_n)-n(x)|^2\,\mathrm{d}x\mathrm{d}t\right)$$
$$=\iint_{\Omega_\sigma}|u(a,\phi)-m(x)|^2\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_\sigma}|v(c,\varphi)-n(x)|^2\,\mathrm{d}x\mathrm{d}t.$$

Proof. For the choice of the parameters $(a,c,\phi,\varphi)\in\mathcal{M}_1^2\times\mathcal{M}_2^2$, multiply the first and second equation of (3.1.1) by ψ and ψb respectively and combine them to get

$$\int_\Omega\left(u_t(a,\phi)\psi+v_t(c,\varphi)\widehat{\psi}\right)\,\mathrm{d}x+\int_\Omega\left(\nabla u(a,\phi)\cdot\nabla\psi+\nabla v(c,\varphi)\cdot\nabla\widehat{\psi}\right)\,\mathrm{d}x$$
$$+\int_\Omega\left(a(x)u(a,\phi)\psi+c(x)v(c,\varphi)\widehat{\psi}\right)\,\mathrm{d}x$$

$$+\int_\Omega\left(b(x)v(c,\varphi)\psi+d(x)u(a,\phi)\widehat{\psi}\right)\,\mathrm{d}x=0, \quad (3.14)$$

where $\psi,\widehat{\psi}$ are the test functions in $H_0^1(\Omega)$. By taking $\psi=u(a,\phi)$, $\widehat{\psi}=v(c,\varphi)$

and integrating with respect to t, we get

$$\int_\Omega\left(|u(a,\phi)|^2+|v(c,\varphi)|^2\right)\,\mathrm{d}x+\iint_{\Omega_T}\left(|\nabla u(a,\phi)|^2+|\nabla v(c,\varphi)|^2\right)\,\mathrm{d}x\mathrm{d}t$$
$$\leq C\iint_{\Omega_T}\left(|u(a,\phi)|^2+|v(c,\varphi)|^2\right)\,\mathrm{d}x\mathrm{d}t+\int_\Omega\left(|\phi|^2+|\varphi|^2\right)\,\mathrm{d}x, \quad (3.15)$$

5) for any t ∈ [0, T]. The above equation together with the assumption on the admissible set implies that the sequences $\{u(a_n,\phi_n),v(c_n,\varphi_n)\}$ are bounded in the space $L^2(0,T;H_0^1(\Omega))^2$. There exists a subsequence, still denoted by $\{u(a_n,\phi_n),v(c_n,\varphi_n)\}$, such that

$$\left.\begin{array}{l} u(a_n,\phi_n)\rightharpoonup\bar{u}\text{ weakly in }L^2(0,T;H_0^1(\Omega)),\\ v(c_n,\varphi_n)\rightharpoonup\bar{v}\text{ weakly in }L^2(0,T;H_0^1(\Omega)).\end{array}\right\} \quad (3.16)$$

Next we have to prove that the converging point is $(\bar{u}, \bar{v}) = (u(a, \phi), v(c, \varphi))$.

In order to prove that, choose a function χ(t) ∈ C 1 [0, T] with the property that χ(T) = 0. Multiply (3.2.4) by χ(t) for $\chi(t)$ for $\{u(a_n, \phi_n), v(c_n, \varphi_n)\}$ and integrate over to

$$\iint_{\Omega_T}\Big[\Big(\nabla u(a_n,\phi_n)\cdot\nabla\psi+\nabla v(c_n,\varphi_n)\cdot\nabla\widehat{\psi}\Big)+\Big(u_t(a_n,\phi_n)\psi+v_t(c_n,\varphi_n)\widehat{\psi}\Big)\Big]\,\chi\,\mathrm{d}x\mathrm{d}t$$

$$+\iint_{\Omega_T}\Big[\Big(a_n u(a_n,\phi_n)\psi+c_n v(c_n,\varphi_n)\widehat{\psi}\Big)+\Big(bv(c_n,\varphi_n)\psi+du(a_n,\phi_n)\widehat{\psi}\Big)\Big]\,\chi\,\mathrm{d}x\mathrm{d}t=0.$$

After some manipulation along with (3.2.5)-(3.2.6), (3.2.7) can be rewrite

$$\begin{aligned}&\iint_{\Omega_T}\Big(\nabla\bar{u}\cdot\nabla\psi+\nabla\bar{v}\cdot\nabla\widehat{\psi}\Big)\,\chi(t)\,\mathrm{d}x\mathrm{d}t+\int_{\Omega}\Big[\phi\psi(0)+\varphi\widehat{\psi}(0)\Big]\,\chi(0)\,\mathrm{d}x\\&-\iint_{\Omega_T}\Big(\bar{u}\psi+\bar{v}\widehat{\psi}\Big)\,\chi_t\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}\Big(a\bar{u}\psi+c\bar{v}\widehat{\psi}+b\bar{v}\psi+d\bar{u}\widehat{\psi}\Big)\,\chi(t)\,\mathrm{d}x\mathrm{d}t=0,\end{aligned} \tag{3.17}$$

After some manipulation along with (3.2.5)-(3.2.6), (3.2.7) can be rewrite

$$\begin{aligned}&\iint_{\Omega_T}\Big(\nabla\bar{u}\cdot\nabla\psi+\nabla\bar{v}\cdot\nabla\widehat{\psi}\Big)\,\chi(t)\,\mathrm{d}x\mathrm{d}t+\int_{\Omega}\Big[\phi\psi(0)+\varphi\widehat{\psi}(0)\Big]\,\chi(0)\,\mathrm{d}x\\&-\iint_{\Omega_T}\Big(\bar{u}\psi+\bar{v}\widehat{\psi}\Big)\,\chi_t\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}\Big(a\bar{u}\psi+c\bar{v}\widehat{\psi}+b\bar{v}\psi+d\bar{u}\widehat{\psi}\Big)\,\chi(t)\,\mathrm{d}x\mathrm{d}t=0,\end{aligned} \tag{3.18}$$

Which is valid for any $\chi(t) \in [0, T]$ with $\chi(T) = 0$. Hence we

$$\begin{aligned}&\iint_{\Omega_T}\Big(\nabla\bar{u}\cdot\nabla\psi+\nabla\bar{v}\cdot\nabla\widehat{\psi}\Big)\,\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}\Big(\bar{u}_t\psi+\bar{v}_t\widehat{\psi}\Big)\,\mathrm{d}x\mathrm{d}t\\&+\iint_{\Omega_T}\Big(a\bar{u}\psi+c\bar{v}\widehat{\psi}+b\bar{v}\psi+d\bar{u}\widehat{\psi}\Big)\,\mathrm{d}x\mathrm{d}t=0,\qquad\forall\psi,\widehat{\psi}\in H_0^1(\Omega),\end{aligned} \tag{3.19}$$

and $\bar{u}(0) = \phi, \bar{v}(0) = \varphi$. ϕ. Therefore, by the definition of $u(a, \phi)$ and $v(c, \varphi)$, us $\bar{u} = u(a, \phi), \bar{v} = v(c, \varphi)$.
Now we are ready to prove the lemma. The equation (3.2.4) can be rewritten as follows

$$\int_\Omega \left((u_n - m)_t \psi + (v_n - n)_t \widehat{\psi}\right) \mathrm{d}x + \int_\Omega \left(\nabla(u_n - m) \cdot \nabla\psi + \nabla(v_n - n) \cdot \nabla\widehat{\psi}\right) \mathrm{d}x$$
$$+ \int_\Omega \left(a_n(u_n - m)\psi + c_n(v_n - n)\widehat{\psi} + b(v_n - n)\psi + d(u_n - m)\widehat{\psi}\right) \mathrm{d}x$$

$$= \int_\Omega \left(\nabla m \cdot \nabla\psi + \nabla n \cdot \nabla\widehat{\psi}\right) \mathrm{d}x - \int_\Omega \left(a_n m\psi + c_n n\widehat{\psi} + bn\psi + dm\widehat{\psi}\right) \mathrm{d}x,$$

$$+ \int_\Omega \left(\nabla m \cdot \nabla(u_n - m) + \nabla n \cdot \nabla(v_n - n)\right) \mathrm{d}x. \quad (3.20)$$

Similar relations hold for u and v for the choice of $(\psi, \widehat{\psi}) = (u - m, v - n)$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(|u - m|^2 + |v - n|^2\right) \mathrm{d}x + \int_\Omega \left(|\nabla(u - m)|^2 + |\nabla(v - n)|^2\right) \mathrm{d}x$$
$$+ \int_\Omega \left(a|u - m|^2 + c|v - n|^2\right) \mathrm{d}x + \int_\Omega (b + d)(v - n)(u - m)\,\mathrm{d}x$$

$$= -\int_\Omega \left(bn(u - m) + dm(v - n)\right) \mathrm{d}x - \int_\Omega \left(am(u - m) + cn(v - n)\right) \mathrm{d}x$$

$$+ \int_\Omega \left(\nabla m \cdot \nabla(u - m) + \nabla n \cdot \nabla(v - n)\right) \mathrm{d}x, \quad (3.21)$$

$$\mathcal{K}_1 = \int_\Omega [u(a - a_n)(u_n - u) + v(c - c_n)(v_n - v)]\,\mathrm{d}x + \int_\Omega (b + d)\,(u - u_n)(v_n - v)\,\mathrm{d}x$$
$$+ \int_\Omega \left[(a_n - a)|u - m|^2 + (c_n - c)|v - n|^2\right] \mathrm{d}x.$$

Integrating over (0, t), for any t ≤ T, we get

$$\|u_n - u\|^2_{L^2(\Omega)} + \|v_n - v\|^2_{L^2(\Omega)} \le 2\int_0^T \mathcal{K}_1 \mathrm{d}t + \|\phi_n - \phi\|^2_{L^2(\Omega)} + \|\varphi_n - \varphi\|^2_{L^2(\Omega)}.$$

By the weak convergence of $\{u(a_n, \phi_n), v(c_n, \varphi_n)\}$ and the assumed convergence on $\{a_n, c_n, \phi_n, \varphi_n\}$, it is easy to show that

$$\int_0^T \mathcal{K}_1 \mathrm{d}t + \|\phi_n - \phi\|^2_{L^2(\Omega)} + \|\varphi_n - \varphi\|^2_{L^2(\Omega)} \to 0 \quad as \quad n \to \infty.$$

Thus we have

$$\max_{t\in[0,T]} \|u_n - u\|^2_{L^2(\Omega)} + \max_{t\in[0,T]} \|v_n - v\|^2_{L^2(\Omega)} \to 0 \quad as \quad n \to \infty. \tag{3.22}$$

Now the desired convergence of the lemma follows immediately by integrating (3.2.13) over [T − σ, T] and using (3.2.14) together with the Poincare inequality. In a similar manner, we can easily prove the second half of the Lemma 3.2.2.

Theorem 3.2.1 There exists at least one minimize to the optimization problem (3.2.1)-(3.2.3).

Proof. Using Lemma 3.2.2, we can easily prove the existence of the minimize (see for instance).

Remark 3.2.1: It has been shown that there has been a minimum which depends on the regularization parameters oscillating 1, oscillating and oscillating the sub region and final observations in cost-functional areas are used. Even though both 5-01, änder2 and Ś have regularization roles, the introduction in cost function of parameter ±1, ±2, in case of an original reverse problem, leads to an approximate optimisation problem. The relationship between −1(), −2 and ¨ must therefore also be considered. There are several ways to select regularization parameters, for instance via the L-curve method, but further details are not covered by a discussion on the optimized selection of these parameters).

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