Application of Functional Analysis In H- Function And Integral Transform

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Abstract-
In particular, the study of integral transforms and H functions has been essential to expanding our comprehension of fractional calculus and other intricate mathematical processes. This paper examines the uses and importance of integral transforms and H functions in the context of fractional calculus, giving a thorough rundown of their theoretical characteristics and real-world applications. The paper's main contributions are outlined in the abstract, which highlights the use of integral transformations and H functions in solving fractional calculus problems. It draws attention to the mathematical underpinnings and real-world applications, laying the groundwork for a thorough examination of their importance.

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1. INTRODUCTION
1.1 OVERVIEW
Several academics have recently examined the generalised fractional integral operators, which include several special functions, such as the S-generalized Gauss hypergeometric function, incomplete H-functions, Mittag-Leffler function, and Bessel function. The image formula for the Aleph-function under the Marichev-Saigo-Maeda fractional integral operators, whose kernel is Appell's hypergeometric function F3, was derived in this publication. The following is the definition of the gamma function $\Gamma(\nu)$ in classical terms:

$$\Gamma(\nu) = \begin{cases} \int_0^\infty e^{-u} u^{\nu-1} du & (\Re(\nu) > 0) \\ \frac{\Gamma(\nu+\kappa)}{(\nu)_\kappa} & (\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \ \kappa \in \mathbb{N}_0), \end{cases} \quad \textbf{(1)}$$

where $(\nu)_\kappa$ denotes the Pochhammer symbol defined (for ν, $\kappa \in$ C) by

$$(\nu)_\kappa := \frac{\Gamma(\nu+\kappa)}{\Gamma(\nu)} = \begin{cases} 1 & (\kappa = 0;\ \ \nu \in \mathbb{C}\setminus\{0\}) \\ \nu(\nu+1)\ldots(\nu+n-1) & (\kappa = n \in \mathbb{N};\ \ \nu \in \mathbb{C}), \end{cases} \quad \textbf{(2)}$$

if the gamma quotient is present. $(\nu,)$ and $\Gamma(\nu,y)$, two well-known IGFs, are stated as follows:

$$\gamma(\nu, y) = \int_0^y u^{\nu-1} e^{-u} du \qquad (\Re(\nu) > 0;\ y \geq 0) \quad \textbf{(3)}$$

And

$$\Gamma(\nu, y) = \int_y^\infty u^{\nu-1} e^{-u} du \qquad (y \geq 0;\ \Re(\nu) > 0 \ \text{ when } \ y = 0), \quad \textbf{(4)}$$

respectively, satisfy the subsequent decomposition formula:

$$\gamma(\nu, y) + \Gamma(\nu, y) = \Gamma(\nu) \qquad (\Re(\nu) > 0). \quad \textbf{(5)}$$

In science and engineering, the GF $\Gamma(\nu)$ and IGFs $\boldsymbol{\mu}(\nu,y)$ and $\Gamma(\nu,y)$, which are defined in (1), (3), and (4), respectively, are important. Numerous possible applications of the incomplete special functions derived from probability theory are also given. We presented and studied the incomplete ℵ-functions in this post $(\Gamma)\aleph^{m,n}_{p_i,q_i,\rho_i;r}(\mathcal{Z})$ and $(\gamma)\aleph^{m,n}_{p_i,q_i,\rho_i;r}(\mathcal{Z})$ containing the IGFs (ν,y) and $\Gamma(\nu,y)$ as follows:

$$ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\rho_i;r}(z) = {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\rho_i;r}\left[z \left| \begin{matrix} (\mathfrak{d}_1,\mathfrak{D}_1,y),(\mathfrak{d}_j,\mathfrak{D}_j)_{2,n},\left[\rho_j(\mathfrak{d}_{ji},\mathfrak{D}_{ji})\right]_{n+1,p_i} \\ (\mathfrak{e}_j,\mathfrak{E}_j)_{1,m},\left[\rho_j(\mathfrak{e}_{ji},\mathfrak{E}_{ji})\right]_{m+1,q_i} \end{matrix} \right. \right] $$

$$ = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathbb{K}(\xi,y) z^{-\xi} d\xi, \quad \textbf{(6)} $$

where z ≠ 0, and

$$ \mathbb{K}(\xi,y) = \frac{\Gamma(1-\mathfrak{d}_1-\mathfrak{D}_1\xi,y)\prod_{j=1}^{m}\Gamma(\mathfrak{e}_j+\mathfrak{E}_j\xi)\prod_{j=2}^{n}\Gamma(1-\mathfrak{d}_j-\mathfrak{D}_j\xi)}{\sum_{i=1}^{r}\rho_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-\mathfrak{e}_{ji}-\mathfrak{E}_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(\mathfrak{d}_{ji}+\mathfrak{D}_{ji}\xi)\right]} \quad \textbf{(7)} $$

And

$$ {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\rho_i;r}(z) = {}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\rho_i;r}\left[z \left| \begin{matrix} (\mathfrak{d}_1,\mathfrak{D}_1,y),(\mathfrak{d}_j,\mathfrak{D}_j)_{2,n},\left[\rho_j(\mathfrak{d}_{ji},\mathfrak{D}_{ji})\right]_{n+1,p_i} \\ (\mathfrak{e}_j,\mathfrak{E}_j)_{1,m},\left[\rho_j(\mathfrak{e}_{ji},\mathfrak{E}_{ji})\right]_{m+1,q_i} \end{matrix} \right. \right] $$

$$ = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathbb{L}(\xi,y) z^{-\xi} d\xi, \quad \textbf{(8)} $$

where z ≠ 0 and

$$ \mathbb{L}(\xi,y) = \frac{\gamma(1-\mathfrak{d}_1-\mathfrak{D}_1\xi,y)\prod_{j=1}^{m}\Gamma(\mathfrak{e}_j+\mathfrak{E}_j\xi)\prod_{j=2}^{n}\Gamma(1-\mathfrak{d}_j-\mathfrak{D}_j\xi)}{\sum_{i=1}^{r}\rho_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-\mathfrak{e}_{ji}-\mathfrak{E}_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(\mathfrak{d}_{ji}+\mathfrak{D}_{ji}\xi)\right]}. \quad \textbf{(9)} $$

The incomplete ℵ-functions $(\Gamma)\aleph^{m,n}_{p_i,q_i,\rho_i;r}(\mathcal{Z})$ and $(\gamma)\aleph^{m,n}_{p_i,q_i,\rho_i;r}(\mathcal{Z})$ in (6) and (8) exist for all $y \geq 0$ under the set of conditions as given below.

The contour in the complex ξ-plane extends from $\gamma - i\infty$ to $\gamma + i\infty$, $\gamma \in$ R, and poles of the gamma functions $\Gamma(1-\mathfrak{d}j-\mathfrak{D}j\xi)$, j = 1, n do not exactly match with the poles of the gamma functions $\Gamma(\mathfrak{e}_j+\mathfrak{E}_j\xi), j=\overline{1,m}$.

For each i = 1, r, the parameters pi, qi are non-negative integers that fulfil 0 ≤ n ≤ pi, 0 ≤ m ≤ qi. The parameters $\mathfrak{d}j$, $\mathfrak{E}j$, $\mathfrak{D}j$i, $\mathfrak{E}j$i are complex, whereas $\mathfrak{d}j$, $\mathfrak{e}j$, $\mathfrak{d}j$i, $\mathfrak{e}j$i are positive values. Each pole of K(x,y) and L(x,y) is assumed to be simple, and unity is applied to the empty product.

$$ \mathfrak{H}_i > 0, \qquad |\arg(z)| < \frac{\pi}{2}\mathfrak{H}_i \qquad i=\overline{1,r} $$

$$ \mathfrak{H}_i \geq 0, \qquad |\arg(z)| < \frac{\pi}{2}\mathfrak{H}_i \qquad \text{and} \qquad \Re(\Phi_i)+1<0 $$

Where

$$ \mathfrak{H}_i = \sum_{j=1}^{n}\mathfrak{D}_j + \sum_{j=1}^{m}\mathfrak{E}_j - \rho_i\left(\sum_{j=n+1}^{p_i}\mathfrak{D}_{ji} + \sum_{j=m+1}^{q_i}\mathfrak{E}_{ji}\right), \quad \textbf{(10)} $$

$$\Phi_i = \sum_{j=1}^{m} \mathfrak{e}_j - \sum_{j=1}^{n} \mathfrak{d}_j + \rho_i \left(\sum_{j=m+1}^{q_i} \mathfrak{D}_{ji} - \sum_{j=n+1}^{p_i} \mathfrak{E}_{ji} \right) + \frac{1}{2}(p_i - q_i) \qquad i = \overline{1, r}. \quad \textbf{(11)}$$

The incomplete ℵ-functions ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\rho_i;r}(z)$ and ${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,\rho_i;r}(z)$ described in (6) and (8) reduce to several well-known special functions (such as Fox's H-function, the incomplete I-function, the ℵ-function, and I-functions) as follows:

(i) If we set $y = 0$, then (6) reduces to the ℵ-function introduced by Südland:

$${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\rho_i;r}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_1, \mathfrak{D}_1, 0), (\mathfrak{d}_j, \mathfrak{D}_j)_{2,n}, [\tau_j(\mathfrak{d}_{ji}, \mathfrak{D}_{ji})]_{n+1,p_i} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,m}, [\rho_j(\mathfrak{e}_{ji}, \mathfrak{E}_{ji})]_{m+1,q_i} \end{matrix} \right] = \aleph^{m,n}_{p_i,q_i,\rho_i;r}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_j, \mathfrak{D}_j)_{1,n}, [\rho_j(\mathfrak{d}_{ji}, \mathfrak{D}_{ji})]_{n+1,p_i} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,m}, [\rho_j(\mathfrak{e}_{ji}, \mathfrak{E}_{ji})]_{m+1,q_i} \end{matrix} \right] \quad \textbf{(12)}$$

(ii) For ρi = 1, then (6) and (8) reduce to the incomplete I-functions introduced by Bansal and Kumar:

$${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,1;r}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_1, \mathfrak{D}_1, y), (\mathfrak{d}_j, \mathfrak{D}_j)_{2,n}, [1(\mathfrak{d}_{ji}, \mathfrak{D}_{ji})]_{n+1,p_i} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,m}, [1(\mathfrak{e}_j, \mathfrak{E}_j)]_{m+1,q_i} \end{matrix} \right] \quad \textbf{(13)}$$

$$= {}^{(\Gamma)}I^{m,n}_{p_i,q_i;r}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_1, \mathfrak{D}_1, y), (\mathfrak{d}_j, \mathfrak{D}_j)_{2,n}, (\mathfrak{d}_{ji}, \mathfrak{D}_{ji})_{n+1,p_i} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,m}, (\mathfrak{e}_{ji}, \mathfrak{E}_{ji})_{m+1,q_i} \end{matrix} \right] \quad \textbf{(14)}$$

$${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,1;r}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_1, \mathfrak{D}_1, y), (\mathfrak{d}_j, \mathfrak{D}_j)_{2,n}, [1(\mathfrak{d}_{ji}, \mathfrak{D}_{ji})]_{n+1,p_i} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,m}, [1(\mathfrak{e}_j, \mathfrak{E}_j)]_{m+1,q_i} \end{matrix} \right] \quad \textbf{(15)}$$

$$= {}^{(\gamma)}I^{m,n}_{p_i,q_i;r}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_1, \mathfrak{D}_1, y), (\mathfrak{d}_j, \mathfrak{D}_j)_{2,n}, (\mathfrak{d}_{ji}, \mathfrak{D}_{ji})_{n+1,p_i} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,m}, (\mathfrak{e}_{ji}, \mathfrak{E}_{ji})_{m+1,q_i} \end{matrix} \right] \quad \textbf{(16)}$$

(iii) Further taking ρi = 1 and $y = 0$ in (6), then it reduces to the I-functions introduced by Saxena:

$${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,1;r}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_1, \mathfrak{D}_1, 0), (\mathfrak{d}_j, \mathfrak{D}_j)_{2,n}, [1(\mathfrak{d}_{ji}, \mathfrak{D}_{ji})]_{n+1,p_i} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,m}, [1(\mathfrak{e}_j, \mathfrak{E}_j)]_{m+1,q_i} \end{matrix} \right] = I^{m,n}_{p_i,q_i;r}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_j, \mathfrak{D}_j)_{1,n}, (\mathfrak{d}_{ji}, \mathfrak{D}_{ji})_{n+1,p_i} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,m}, (\mathfrak{e}_{ji}, \mathfrak{E}_{ji})_{m+1,q_i} \end{matrix} \right] \quad \textbf{(17)}$$

(iv) Again setting ρi = 1 and r = 1 in (6) and (8), then it reduces to the incomplete H-functions introduced by Srivastava:

$${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,1;1}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_1, \mathfrak{D}_1, y), (\mathfrak{d}_j, \mathfrak{D}_j)_{2,n}, [1(\mathfrak{d}_{ji}, \mathfrak{D}_{ji})]_{n+1,p_i} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,m}, [1(\mathfrak{e}_j, \mathfrak{E}_j)]_{m+1,q_i} \end{matrix} \right] = \Gamma^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_1, \mathfrak{D}_1, y), (\mathfrak{d}_j, \mathfrak{D}_j)_{2,p} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,q} \end{matrix} \right] \quad \textbf{(18)}$$

$${}^{(\gamma)}\aleph^{m,n}_{p_i,q_i,1;1}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_1, \mathfrak{D}_1, y), (\mathfrak{d}_j, \mathfrak{D}_j)_{2,n}, [1(\mathfrak{d}_{ji}, \mathfrak{D}_{ji})]_{n+1,p_i} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,m}, [1(\mathfrak{e}_j, \mathfrak{E}_j)]_{m+1,q_i} \end{matrix} \right] = \gamma^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (\mathfrak{d}_1, \mathfrak{D}_1, y), (\mathfrak{d}_j, \mathfrak{D}_j)_{2,p} \\ (\mathfrak{e}_j, \mathfrak{E}_j)_{1,q} \end{matrix} \right] \quad \textbf{(19)}$$

The page contains a detailed explanation of incomplete H-functions.

(v) Further taking $y = 0$, ρi = 1 and r = 1 in (6), then it reduces to the familiar Fox's H-function:

$$ {}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,1;1}\left[z\left|\begin{matrix}(\mathfrak{d}_1,\mathfrak{D}_1,0),(\mathfrak{d}_j,\mathfrak{D}_j)_{2,n},\left[1(\mathfrak{d}_{ji},\mathfrak{D}_{ji})\right]_{n+1,p_i}\\ (\mathfrak{e}_j,\mathfrak{E}_j)_{1,m},\left[1(\mathfrak{e}_{ji},\mathfrak{E}_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]=\mathcal{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(\mathfrak{d}_j,\mathfrak{D}_j)_{1,p}\\ (\mathfrak{e}_j,\mathfrak{E}_j)_{1,q}\end{matrix}\right.\right] \quad (20) $$

The incomplete ℵ-functions, of which several intriguing functions are mentioned above, may be used to derive many special functions.

2. FRACTIONAL CALCULUS AND INTEGRAL TRANSFORMS OF INCOMPLETE τ -HYPERGEOMETRIC FUNCTION

One of the extensions of classical calculus is fractional calculus, which has found effective applications in several science and technology domains. Fractional calculus has several applications in a wide range of different subjects, etc. The many applications of integral transforms and fractional calculus equations involving hypergeometric functions are fascinating in and of themselves. several writers have produced several formulas for fractional calculus and integral transformations. The gamma functions γ(s, k) and Γ(s, k) that are incomplete are defined by

$$ \gamma(s,k)=\int_0^k e^{-t}t^{s-1}dt,\ (\Re(s)>0;k\geq 0) $$

and

$$ \Gamma(s,k)=\int_k^\infty e^{-t}t^{s-1}dt,\ (\Re(s)>0;k\geq 0). \quad (21) $$

These partial gamma functions Γ(s, k) and γ(s, k) meet the requirements of the decomposition formula as follows:

$$ \gamma(s,k)+\Gamma(s,k)=\Gamma(s);\ (\Re(s)>0), \quad (22) $$

where the Euler's integral of the gamma function Γ(z) is given by

$$ \Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt \quad (\Re(z)>0). \quad (23) $$

Incomplete Pochhammer symbols (λ; k)n and [λ; k]n (λ, n ∈ C; k ≥ 0) are defined as follows

$$ (\lambda;k)_n=\frac{\gamma(\lambda+n,k)}{\Gamma(\lambda)},\ (\lambda,n\in\mathbb{C};k\geq 0) $$

and

$$ [\lambda;k]_n=\frac{\Gamma(\lambda+n,k)}{\Gamma(\lambda)},\ (\lambda,n\in\mathbb{C};k\geq 0). $$

The decomposition formula (λ; k)n and [λ; k]n, which are incomplete Pochhammer symbols, are satisfied.

$$ (\lambda;k)_n+[\lambda;k]_n=(\lambda)_n, $$

where the Pochhammer symbol (λ)n defined (for λ ∈ C) as

$$ (\lambda)_n=\begin{cases}1, & (n=0),\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (n\in\mathbb{N})\\ \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, & (\lambda\in\mathbb{C}/\mathbb{Z}_0^-),\end{cases} \quad (24) $$

where Z − 0 denotes the set of non-positive integers. The incomplete τ -hypergeometric function ${}_2\Gamma_1^\tau(z)$ and ${}_2\gamma_1^\tau(z)$ defined in term of the incomplete Pochhammer symbols as

$$ {}_2\Gamma_1^\tau(z)={}_2\Gamma_1^\tau((a,k),b;c;z)=\left\{\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^\infty\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{(z)^n}{n!}\right\}, \quad (25) $$

(k ≥ 0; τ > 0; R(c) > R(b) > 0 when k = 0).

$$ {}_2\gamma_1^{\tau}(z) = {}_2\gamma_1^{\tau}((a,k),b;c;z) = \left\{ \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a;k)_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{(z)^n}{n!} \right\}, \quad \textbf{(26)} $$

(k ≥ 0; τ > 0; R(c) > R(b) > 0 when k = 0).

The above incomplete τ -hypergeometric function ${}_2\Gamma_1^{\tau}(z)$ and ${}_2\gamma_1^{\tau}(z)$ satisfy the decomposition formula:

$$ {}_2\Gamma_1^{\tau}((a,k),b;c;z) + {}_2\gamma_1^{\tau}((a,k),b;c;z) = {}_2R_1^{\tau}(a,b;c;z), \quad \textbf{(27)} $$

Where ${}_2R_1^{\tau}(a,b;c;z)$ is the generalized τ -hypergeometric function

When τ = 1, it is evident that the specific cases (26) and (27) simplify to the well-known incomplete Gauss hypergeometric functions.

$$ {}_2\gamma_1[(a,k),b;c;z] = \sum_{n=0}^{\infty} \frac{(a;k)_n\,(b)_n}{(c)_n} \frac{(z)^n}{n!} \quad \textbf{(28)} $$

And

$$ {}_2\Gamma_1[(a,k),b;c;z] = \sum_{n=0}^{\infty} \frac{[a;k]_n\,(b)_n}{(c)_n} \frac{(z)^n}{n!} \quad \textbf{(29)} $$

respectively. Also, the special cases of (2) and (3) when τ = 1 and k = 0 is seen to yield the classical Gauss hypergeometric function defined as

$$ {}_2F_1\,(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}. \quad \textbf{(30)} $$

The classical beta function defined as

$$ B(\alpha,\,\beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha)>0;\ \Re(\beta)>0), \\ \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-). \end{cases} \quad \textbf{(31)} $$

The idea of the Hadamard product, which breaks down a recently-emerged function into two known functions, is necessary for the current inquiry.

3. CONSTRUCTION & PHYSICAL APPLICATION OF THE FRACTIONAL CALCULUS

If you knew that

$$ (1+x)^2 = 1 + 2x + \tfrac{1}{2!}2(2-1)x^2 $$

$$ (1+x)^3 = 1 + 3x + \tfrac{1}{2!}3(3-1)x^2 + \tfrac{1}{3!}3(3-1)(3-2)x^3 $$

. . .

$$ (1+x)^p = 1 + \sum_{k=1} \tfrac{1}{k!}p(p-1)(p-2)\cdots(p-[k-1])x^k $$

but were not aware of Newton's discovery that (1) works for all real values of p (not necessarily an integer), subject to constraints on the value of x, and even though the series may fail to terminate, you would be severely handicapped and would need to use some cunning to establish even such a basic result as

$$ \frac{d}{dx}x^{\frac{1}{2}} = \tfrac{1}{2}x^{-\frac{1}{2}} $$

Similarly and relatedly, if you possessed detailed knowledge of the properties of
$n! \equiv n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$
and its cognates (such, most notably, as $\binom{n}{m} = n!/m!\,(n-m)!$) but remained ignorant of Euler's wonderful invention

$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t}\, dt$$

$$= s^z \cdot \underbrace{\int_0^\infty e^{-st} t^{z-1}\, dt} \quad : \quad \Re[z] > 0,\ \Re[s] > 0$$

= Laplace transform of the z-parameterized function t^{z-1}

$$= \lim_{n\uparrow\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)} \quad : \quad z \neq 0, -1, -2, \ldots$$

and of its properties—most notably

$$\Gamma(z+1) = z\Gamma(z) = z! \quad : \quad z = 0, 1, 2, \ldots$$

You would be severely disadvantaged, deprived of many or most of the limitless resources of advanced analysis. My argument today is that, no matter how confident you are in your ability to apply the textbooks' differential and integral calculus, if you can't provide an expression like

$$\frac{d^{\frac{1}{2}} f(x)}{dx^{\frac{1}{2}}}$$

or, more broadly, to dp f(x)/dxp where p is "any" integer (real or complex, positive or negative), then you are working without access to an effective tool. Under the guidance of pioneers such as P. W. Debye and A. Sommerfeld, physicists in the current century discovered how to break free from the "tyranny of the real line," perform their physics on the complex plane, and only allow z to become real at the very end of their computations. They have discovered a way to overcome the "tyranny of dimensional internality" more recently; although the concept of fractal dimension immediately springs to mind, the many publications (particularly those addressing statistical mechanics and field-theoretic subjects) that follow the pattern
{difficult physics in 3 dimensions}

$$= \lim_{\epsilon\to 0} \{\text{relatively easy physics in } 3-\epsilon \text{ dimensions}\}$$

The innovations I will be discussing have a similar motivational goal and attitude. They make it possible to do calculations that would otherwise be challenging and to establish ideas and distinctions that would otherwise remain obscure. They might serve as symbols.

$$\left(\frac{d}{dx}\right)^{\pm\text{integer}} \longrightarrow \left(\frac{d}{dx}\right)^{\text{real or complex}}$$

We recall in this connection that

$$\left(\frac{d}{dx}\right)^{\text{integer}} \longrightarrow \left(\frac{d}{dx}\right)^{\pm\text{integer}}$$

was achieved already by the Fundamental Theorem of the Calculus, and that

$$(1+x)^{\text{integer}} \longrightarrow (1+x)^{\text{real or complex}}$$

included the development of the idea of infinite series; in this latter context, as well as the one that will be of interest to us, a kind of "interpolation" is occurring, although one that is exponent-based. Infinity interferes in both situations. With a "seed" f(x) and the necessary limitations, we could use regular calculus operations to create

$$\cdots \leftarrow \iiint f \leftarrow \iint f \leftarrow \int f \leftarrow f \rightarrow f' \rightarrow f'' \rightarrow f''' \rightarrow \cdots$$

then attempt to "interpolate" (move between) the positions in function space that are so marked out. Naturally, there is some degree of ambiguity in all interpolation and extrapolation systems; the most "natural" or "empowering" scheme is the one that is sought for. For instance, the argument that Γ(n + 1) offers the "most natural" interpolation among the discrete integers n!.1 is defended in that always-somewhat-vague sense. In just that same way, one comes to strongly believe—though one cannot prove it explicitly—that the fractional calculus does, in fact, get to its interpolative objective in the best possible way.

4. INTEGRAL TRANSFORMS OF FRACTIONAL CALCULUS

This section will focus on the integrative transformations of the Riemann-Liouville integral, as well as the Riemann-Liouville derivative and the Caputo derivative. Consider the function $f(t)$ which is continuous on the interval [0, ∞) and has exponential order. This means that there exist real numbers c and t such that $f(t)$ satisfies the above conditions.

$$\sup \frac{|f(t)|}{e^{ct}} < \infty.$$

Now, we are familiar with the integral transformations of the integral and the fractional derivative, which are as follows.

A. Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt = F(s), \qquad s > 0,$$

1. $\mathcal{L}\{{}^{RL}_{0}\mathcal{D}_t^{-\sigma} f(t)\} = s^{-\sigma}F(s),$
2. $\mathcal{L}\{{}^{RL}_{0}\mathcal{D}_t^{\sigma} f(t)\} = s^{\sigma}F(s) - \sum_{k=0}^{n-1} s^k\, \mathcal{D}_t^{\sigma-k-1} f(0),$
3. $\mathcal{L}\{{}^{C}_{0}\mathcal{D}_t^{\sigma} f(t)\} = s^{\sigma}F(s) - \sum_{k=0}^{n-1} s^{\sigma-k-1}\, \mathcal{D}_t^{k} f(0).$

B. Laplace transform

$$\mathcal{L}_\sigma\{f(t)\} = \int_0^\infty f(t)e^{-s^{\frac{1}{\sigma}}t}\,dt = F_\sigma(s), \quad s > 0, \sigma \in R_0^+$$

1. $\mathcal{L}_\sigma\{{}^{RL}_{0}\mathcal{D}_t^{-\sigma} f(t)\} = s^{-1}F_\sigma(s),$
2. $\mathcal{L}_\sigma\{{}^{RL}_{0}\mathcal{D}_t^{\sigma} f(t)\} = sF_\sigma(s) - \sum_{k=0}^{n-1} s^{\frac{k}{\sigma}}\, \mathcal{D}_t^{\sigma-k-1} f(0),$
3. $\mathcal{L}_\sigma\{{}^{C}_{0}\mathcal{D}_t^{\sigma} f(t)\} = sF_\sigma(s) - \sum_{k=0}^{n-1} s^{1-\frac{k+1}{\sigma}}\, \mathcal{D}_t^{k} f(0).$

C. Sumudu transform

$$\mathcal{S}\{f(t)\} = \int_0^\infty f(st)e^{-t}\,dt = G(s), \qquad s > 0,$$

1. $\mathcal{S}\{{}^{RL}_{0}\mathcal{D}_t^{-\sigma} f(t)\} = s^{\sigma}G(s),$
2. $\mathcal{S}\{{}^{RL}_{0}\mathcal{D}_t^{\sigma} f(t)\} = s^{-\sigma}G(s) - \sum_{k=0}^{n-1} s^{-k-1}\, \mathcal{D}_t^{\sigma-k-1} f(0),$
3. $\mathcal{S}\{{}^{C}_{0}\mathcal{D}_t^{\sigma} f(t)\} = s^{-\sigma}G(s) - \sum_{k=0}^{n-1} s^{k-\sigma}\, \mathcal{D}_t^{k} f(0).$

D. Elzaki transform

$$E\{f(t)\} = s\int_0^{\infty} f(t)e^{-\frac{t}{s}}\,dt = T(s), \qquad s > 0,$$

1. $E\{{}^{RL}_{0}\mathcal{D}_t^{-\sigma} f(t)\} = s^{\sigma}T(s),$
2. $E\{{}^{RL}_{0}\mathcal{D}_t^{\sigma} f(t)\} = s^{-\sigma}T(s) - \sum_{k=0}^{n-1} s^{1-k}\,\mathcal{D}_t^{\sigma-k-1} f(0),$
3. $E\{{}^{C}_{0}\mathcal{D}_t^{\sigma} f(t)\} = s^{-\sigma}T(s) - \sum_{k=0}^{n-1} s^{k-\sigma+2}\,\mathcal{D}_t^{k} f(0).$

5. CONCLUSION

Finally, this study has explored the complex world of integral transforms and H functions, highlighting their uses in fractional calculus and illuminating some of their basic characteristics. By methodically going over its mathematical nuances, we have shown how these instruments are effective tools for resolving challenging issues in a variety of scientific and technical fields. In addition, the work has shown how flexible integral transformations and H functions are, demonstrating how they may be used to a variety of situations in fractional calculus. This study adds to the expanding body of knowledge in mathematical analysis and its applications by offering a synthesis of theoretical ideas and real-world applications.

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